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Mathematics and Arts: Connections between Theory and Practice in the Medieval Islamic World

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Two mathematical sources, *On the Geometric Constructions Necessary for the Artisan*, by Abu'l-Wafā' (ca. 940–998), and the anonymous work, *On Interlocks of Similar or Corresponding Figures* (ca. 1300), provide us with insight into the collaboration between mathematicians and artisans in the Islamic world. In this paper I present a series of quotations from these two sources, which show that mathematicians taught geometry to artisans by means of cut-and-paste methods and of geometrical figures that had the potential of being used for ornamental purposes. © 2000 Academic Press

Matematikle ilgili iki kaynak bize İslam dünyasında matematikçiler ile sanatkarlar arasındaki işbirliği konusunda aydınlatıcı bilgiler sunuyor. Bu kaynaklardan biri Abu'l-Vefa (ca. 940–998) tarafından yazılan "Sanatkarın ihtiyaç duyduğu geometrik çizimler," diğeri anonim bir yazarın kaleme aldığı "İçiçe geçen benzer veya karşılıklı şekiller" (ca. 1300). Bu iki kaynaktan derlediğim bir dizi alıntıya yer verdiğim bu makalede görüyoruz ki matematikçiler sanatkarlara kes-ve-yapıştır yöntemiyle geometri öğretirken, aynı zamanda önerdikleri geometrik şekillerin bezeme sanatlarında kullanılabilir olmasına özen gösteriyorlardı. © 2000 Academic Press

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INTRODUCTION

Intriguing patterns ornamenting architectural monuments and other objects of art bear witness to the predominance of geometry in Islamic art. Traditionally, the artisans who produced them were believed to be experts in geometry. Recent studies, however, have shown that mathematicians who taught practical geometry to artisans played a decisive role in the creation of those patterns and perhaps in designing the buildings themselves. Medieval Islamic *mathematical* literature is thus our main source for the relation between mathematics and the architectural arts in medieval Islam [20–23; 27; 37–41].

Abu'l-Wafā' al-Būzjānī (940–ca. 998)[1] tells us in his book, *On the Geometric Constructions Necessary for the Artisan* (Kitāb fīmā yaḥtāju ilayhi al-ṣāniʿ min al-aʿmāl al-handasiya; hereinafter *Geometric Constructions*), that he attended meetings between geometers and artisans in Baghdād [4, 53].[2] Such meetings were a widespread phenomenon in the Islamic world. In 11th- century Isfahan, for example, ʿUmar Khayyām solved a right triangle with

[1] On Abu'l-Wafā', see [46, 321–325].

[2] For my study, I have used the manuscript Istanbul, Ayasofya 2753 (a facsimile of this manuscript is in [42]). It was copied in the first half of the 15th century for Ulugh Beg's library in Samarkand and was brought to Istanbul by ʿAlī Kuşçu in 1471, when he settled and started teaching in the Ayasofya Madrasa there. This manuscript is

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the aid of a cubic equation in an untitled treatise. He explained that the treatise was prompted by a question at a meeting, which seems to have been attended by artisans and geometers [12; 31; 34; 43]. In the 15th century, Ghiyāth al-Dīn Jamshīd al-Kāshī solved a problem about a triangular leveling instrument at the construction site of the astronomical observatory in Samarkand during a meeting of artisans, mathematicians, and other dignitaries [29, 101]. We learn from Caʿfer Efendi that meetings between architects and geometers were held frequently in Istanbul in the late 16th and early 17th centuries [16, 28]. Caʿfer Efendi reportedly compiled a treatise from his notes taken at such meetings over a period of 20 years [16, 22–23], although no copies of this treatise have yet been found.

Here, I discuss an anonymous Persian work on ornamental geometry, which may be similar to the lost work by Caʿfer Efendi, and which seems to have been compiled from notes taken by a scribe at a series of meetings between geometers and artisans.[3] The title of the work appears as a vertical marginal note: *Interlocks of Similar or Corresponding Figures* (Fī tadākhul al-ashkāl al-mutashābiha aw al-mutawāfiqa; hereinafter *Interlocks of Figures*) [11]. This work can be dated to around 1300 because the mathematician, Abū Bakr al-Khalīl al-Tājir al-Raṣadī (ca. 1300),[4] is cited twice as one of the participants of the discussions, and the text probably came from Tabrīz.[5]

The mathematicians seemed pleased to be introduced to a field full of rewarding concrete applications. For instance, ʿUmar Khayyām says, "If it were not for the highness of this meeting, may its highness last forever, and for the appropriateness of the proposer of the question, may God bless him, I would have been far away from this field" [12, 336; 43, 90]. It is thus conceivable that some of the aesthetic, spatial, and structural innovations of Islamic architecture are due to mathematicians.[6]

The meetings between mathematicians and artisans were normally not documented, but fortunately there are exceptions. The account by Abu'l-Wafā' in *Geometric Constructions* and the descriptions in *Interlocks of Figures* provide insights into how mathematicians and artisans collaborated. At some sessions, mathematicians gave instructions on certain principles and practices of geometry. At others, they worked on geometric constructions of two- or three-dimensional ornamental patterns or gave advice on the application of geometry to architectural construction. Below, I argue that the mathematicians used cut-and-paste methods as a didactical tool in teaching geometry to artisans. By presenting literal

dedicated to the Buyid ruler Al-Manṣūr Bahā' al-Dawla, who is referred to by the title Shāhinshāh. This title could have been used only for the rulers of Persia, and Al-Manṣūr was the ruler of Persia from A.D. 998 to 1013. Therefore, the text was probably compiled by a student of Abu'l-Wafā' after the death of his master in 998. This argument can be supported by the fact that passages such as "Abu'l-Wafā' (or the professor or the sage) said" appear in the text. For other manuscript copies, commentaries, and modern translations of Abu'l-Wafā's text, see [1–3; 5–8; 14–15; 32–33; 42; 47–48].

[3] In [40], I have argued that this treatise was a compilation of notes taken over a period of time.

[4] According to the information kindly provided by Mohammad Bagheri and Mehran Akhbarifar (both of Tehran), Abū Bakr al-Khalīl is cited as the father of the copyist of a manuscript in Mashhad. Since the manuscript was copied in 1327–1328, Abū Bakr al-Khalīl must have been active around 1300. The only other available information about this mathematician is a remark found at the end of an anonymous undated Persian treatise (in the same Paris manuscript as *Interlocks of Figures*), praising him as "the pride of geometers" [10, 118v].

[5] The provenance of *Interlocks of Figures* and the milieu in which it was written will be discussed below, and in my English translation with commentary, to appear in due course. For modern studies on this work, see [13, 315–340; 15, 73–95; 17–19; 35, 129–181; 40].

[6] In [41], I have argued that ʿUmar Khayyām was the designer of the North Dome Chamber of the Friday Mosque of Isfahan, which is celebrated for the maturity and elegance of its proportions.

translations with modern interpretations of passages from the *Geometric Constructions* and from the *Interlocks of Figures*, I aim to shed light on the exact nature of the collaboration between mathematicians and artisans in the medieval Islamic world.[7]

THE GEOMETRIC CONSTRUCTIONS OF ABU'l-WAFĀ'

In the introduction to Chapter 10, of *Geometrical Constructions*, titled "On Dividing and Composing the Squares," Abu'l-Wafā' noted that artisans widely used the technique of cutting figures. He stated that his chapter aimed to describe rules artisans should follow because they made elementary mistakes in dividing and putting together squares. The artisans who answered this description were probably either carpenters, especially those who were skillful at inlaying, or floor mosaicists whose job involved cutting a single piece of material into parts and arranging those parts in patterns composed of different figures. As the chapter title suggests, the pieces to be cut and the finished patterns were usually in square forms. Indeed, existing floor mosaics and the woodworks that decorate doors, shutters, mimbars, and other objects of art were mostly arranged in square panels, particularly before the 11th century.

Chapter 10 consists of various subsections, such as "On Assembling and Dividing [Unit] Squares If Their Number Is Not the Sum of Two Square Numbers." Abu'l-Wafā' began the latter subsection with a description of the differing attitudes of mathematicians and artisans toward geometry:

> A number of geometers and artisans have erred in the matter of these squares and their assembling. The geometers [have erred] because they have little practice in constructing, and the artisans [have erred] because they lack knowledge of proofs. The reason is that, since the geometers do not have experience in construction, it is difficult for them to approximate [i.e., construct in practice]—in the way required by the artisan—what is known to be correct by proofs by means of lines [i.e., diagrams]. The aim of an artisan is what is approximated by the construction, and [for him] correctness is apparent by what he perceives through his senses and by inspection. He is not concerned with the proofs by means of lines. If, for a geometer, the proof of something is established by way of imagination, he is not concerned with the [apparent] correctness or incorrectness of something by inspection. But we do not doubt that everything that the artisan sees is taken from what the geometer had worked out previously and what had been proved to be correct. Therefore, the artisan and the surveyor take the choice parts [literally, the cream] of the thing, and they do not think about the methods by which correctness is established. Thus occur the errors and mistakes.
>
> The geometer knows the correctness of what he wants by means of proofs, since he is the one who has derived the notions on which the artisan and the surveyor base their work. However, it is difficult for him to transform what he has proved into a [practical] construction, since he has no experience with the practical work of the artisan and the surveyor. If the skillful among these geometers are asked about something in dividing the figures or multiplying the lines, they are confused and need a long time to think. Sometimes they are successful, and it is easy for them; but sometimes it is difficult for them and they do not find its construction. [4, 52–53]

Abu'l-Wafā' thus suggested that surveyors and artisans were considered appliers of practical geometry and that only those who dealt with theoretical geometry were qualified for the title *muhandis* [geometer]. In later centuries, surveyors, too, were called *muhandis*.

[7] I am indebted to Taner Avcı and Zaka Siddiqi for their help in the translations into English of the passages quoted from the Arabic text of *Geometric Constructions* and from the Persian text of *Interlocks of Figures*, respectively. Jan P. Hogendijk provided corrections to those translations. The Arabic text on which the translations in this paper are based is printed in the Appendix. Since I am preparing an edition and English translation of the whole *Interlocks of Figures*, the Persian texts of the quoted passages are not provided here.

Abu'l-Wafā' continued:

> I was present at some meetings in which a group of geometers and artisans participated. They were asked about the construction of a square from three squares. A geometer easily constructed a line such that the square of it is equal to the three squares, but none of the artisans was satisfied with what he had done. The artisan wants to divide those squares into pieces from which one square can be assembled, as we have described for two squares and five squares. [4, 53]

This passage is important, since Abu'l-Wafā' stated explicitly that he took part in several meetings between geometers and artisans. He seemed to have had authority over both parties owing to his expertise in geometry and to his familiarity with the works of artisans. It is thus likely that he was in his 50s at the time of these meetings. As we will see, he understood the needs and problems of the artisans, tried to accommodate these, and encouraged other geometers to do the same. It appears that his book, *Geometric Constructions*, was motivated by such meetings and by his efforts to advance Islamic art. In his book, he displayed knowledge of pure geometry, familiarity with practical applications, and skill in teaching theoretical subjects to practical-minded people.

The question asked at the meeting gives an indication of the nature of the problems discussed. To construct one square out of three may initially seem to be a problem without practical value for the artisans. From the point of view of the design process, however, it is related to the tendency of Islamic architects and artisans to use modular designs. Recently discovered authentic plans of buildings and architectural decorations show that their designs were based on square-grid layouts or radial organization of squares and rhombi [35, 3–27; 36; 28]. The 15th-century mathematician Ghiyāth al-Dīn Jamshīd al-Kāshī noted that muqarnas[8] designs were generated from a unit square which he specifically referred to as "module" (miqyās) [20, 209; 37, 37–39; 38, 57]. It seems that artisans in 10th- century Baghdād followed the same tradition of modular design based on square units when they asked how to obtain a square equal in area to three unit squares. What they really wanted, however, was not a theoretical geometric solution like a geometer would have proposed. Instead, they wanted a nice, practical solution based on the "method of dividing and assembling" (cut-and-paste methods) because such a solution had the potential to inspire an ornamental pattern. Abu'l-Wafā' 's earlier solutions of the same problem for two squares and five squares were indeed very elegant and thus added fine patterns to their repertoire.

The ornamental qualities of these two solutions merit study in greater detail. Abū'l-Wafā' was interested in the general problem of constructing (or, as he put it, assembling) a square equal in area to n unit squares. In the subsection "On Assembling Squares If Their Number Is the Sum of Two Squares," he distinguished two cases: "if the number is the sum of two equal squares" and "if the number is the sum of two unequal squares" [4, 48–50]. Abu'l-Wafā' explained the solution of these cases in general terms, and he then discussed $n = 8$ as an example of the first case, and $n = 13$ and $n = 10$ as examples of the second case. It is likely that he explained solutions for $n = 2$ and $n = 5$ during the discussions, although our text does not mention them explicitly.[9] From Abū'Wafā's general explanations, his solutions are easily deduced.

[8] The word *muqarnas* means a stalactite vault, a three-dimensional decorative element in Islamic architecture.

[9] Perhaps the student who compiled and published these notes after Abu'l-Wafā' 's death missed the explanations concerning $n = 2$ and $n = 5$ during the discussions. See footnote 2.

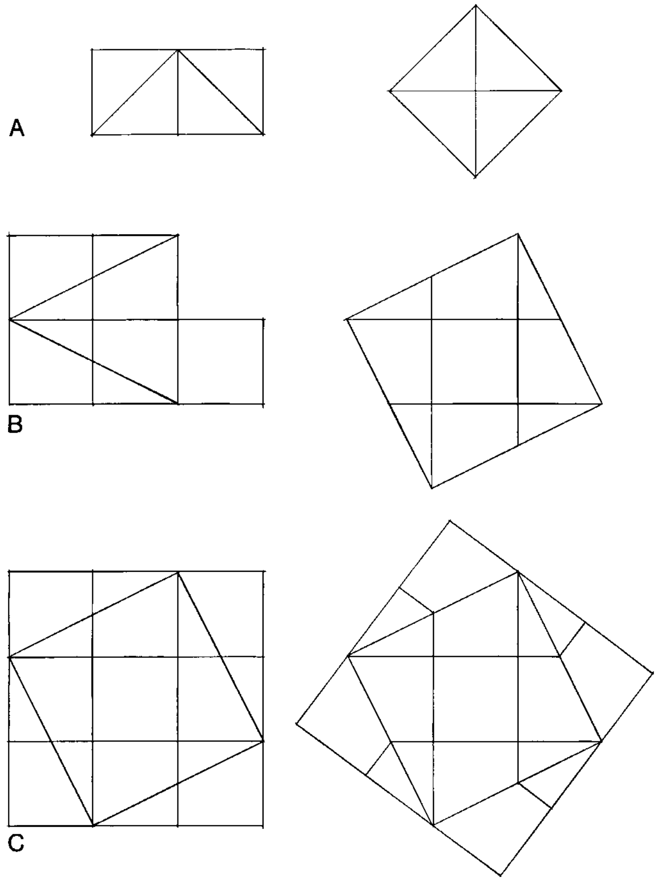


FIGURE 1

To assemble a square from two squares, divide each of the two unit squares along a diagonal, and assemble the four congruent triangles thus obtained by joining their perpendicular sides. Their diagonals become the sides of the required square (Fig. 1A).[10] The pattern created by this construction had been known to artisans since ancient times. Since many examples of it may be found in floor mosaics dating from the Roman and Byzantine periods, the pattern was presumably not new to artisans in 10th-century Baghdād. What probably appealed to them was its simple mathematical formulation by Abu'l-Wafā' and thus its potential for expansion in four directions by simply adding the triangular pieces. A study of Islamic ornaments shows that the most favored patterns are the ones that have the flexibility of being expanded by repetition or by radial enlargement.

To assemble a square from five squares, we have $n = 5 = 1 + 4 = 1^2 + 2^2$. From four unit squares compose two rectangles with length two units and width one unit. Cut the

[10] Abu'l-Wafā' 's general construction for $n = 2m^2$ can be summarized as follows. Cut the $2m^2$ unit squares in halves along a diagonal and arrange the $4m^2$ congruent triangles into a big square consisting of m^2 squares of area two unit squares.

rectangles along their diagonals to obtain four congruent triangles. Assemble the triangles around the remaining unit square by joining the sides around the right angles. Thus, one obtains the required square. The side of the required square is the diagonal of the rectangle (Fig. 1B).[11] The elegance and rotational symmetry of this construction probably attracted the attention of the artisans, who were always looking for new patterns to add to their repertoire. By joining four more congruent right-angled triangles along their hypotenuses to the square equal to five unit squares, one obtains a larger square equal to nine unit squares. The four new congruent triangles can be positioned in two ways. If they are placed so as to form four "almonds"[12] with the four other congruent triangles, a charming dynamic pattern results (Fig. 1C). Numerous versions of this pattern are found on wall surfaces, portals, minarets, doors, chests, etc. from different periods and from different countries. The earliest extant example is from the wooden door of the Mosque of Imām Ibrāhīm in Mosul dating from A.D. 1104 [18, 766]. A fine specimen of this popular motif in Islamic art is a 17th-century tile panel from the western *iwān*[13] of the Friday Mosque in Isfahan (Fig. 2). If the almonds are subdivided into three smaller ones and the pattern is repeated, an interesting composition can be generated and repeated indefinitely to cover whole wall surfaces (Fig. 3). However complicated the composition appeared, the area of the finished work could easily be calculated from the unit square at the middle, thanks to the modular design. This pattern may have appealed to Islamic artisans because of its flexibility and convenience.

The preceding evidence suggests that artisans who attended the meetings described by Abu'l-Wafā' were interested in his solutions because of possible applications in the design of ornaments. Thus, they could have been interested in a solution for $n = 3$ that could be applied in a similar way. On the solutions that they themselves produced, however, Abu'l-Wafā' had much to say:

> The artisans proposed a number of methods, some of which can be proved and others of which are incorrect, but the methods which cannot be proved resemble the truth in appearance, so someone who looks at them may think that they are correct. We shall present these methods so that the correct ones may be distinguished from the false ones, and someone who looks into this subject will not make a mistake by accepting a false method, God willing.
>
> One of the artisans placed one of the squares in the middle and bisected the second by means of the diagonal and placed it [the parts] on both sides of the [first] square. He drew from the center of the third square two straight lines to two of its angles, not on one diagonal, and he drew a line from it [the center] to the midpoint of the side opposite the triangle which is produced by two lines. Then the square is divided into two trapezia and a triangle. Then he put the triangle below the first square and the two trapezia above it, and he joined the two longer sides [of the trapezia] in the middle. Thus he obtained a square as in this figure [Fig. 4].
>
> Abu'l-Wafā' said:[14] But this figure which he constructed is fanciful, and someone who has no experience in the art or in geometry may consider it correct, but if he is informed about it he knows that it is false. [4, 53–54]

[11] Abu'l-Wafā''s general construction for $n = a^2 + b^2$ with $a > b$ can be summarized in this way. From $2ab$ unit squares, compose four right-angled triangles with length a and width b. Then assemble these triangles around a square consisting of $(a - b)^2$ unit squares. If x is the side of the square obtained in this way, $x^2 = (a - b)^2 + 2ab = a^2 + b^2$.

[12] Almond is the general term in Islamic languages for the kite-shaped figure peculiar to Islamic art.

[13] An *iwān* is a vaulted or flat roofed hall open at one end, usually looking into a courtyard.

[14] The appearance of Abu'l-Wafā''s name in the text implies that *Geometric Constructions* was compiled later from notes taken by a student. See footnote 2.

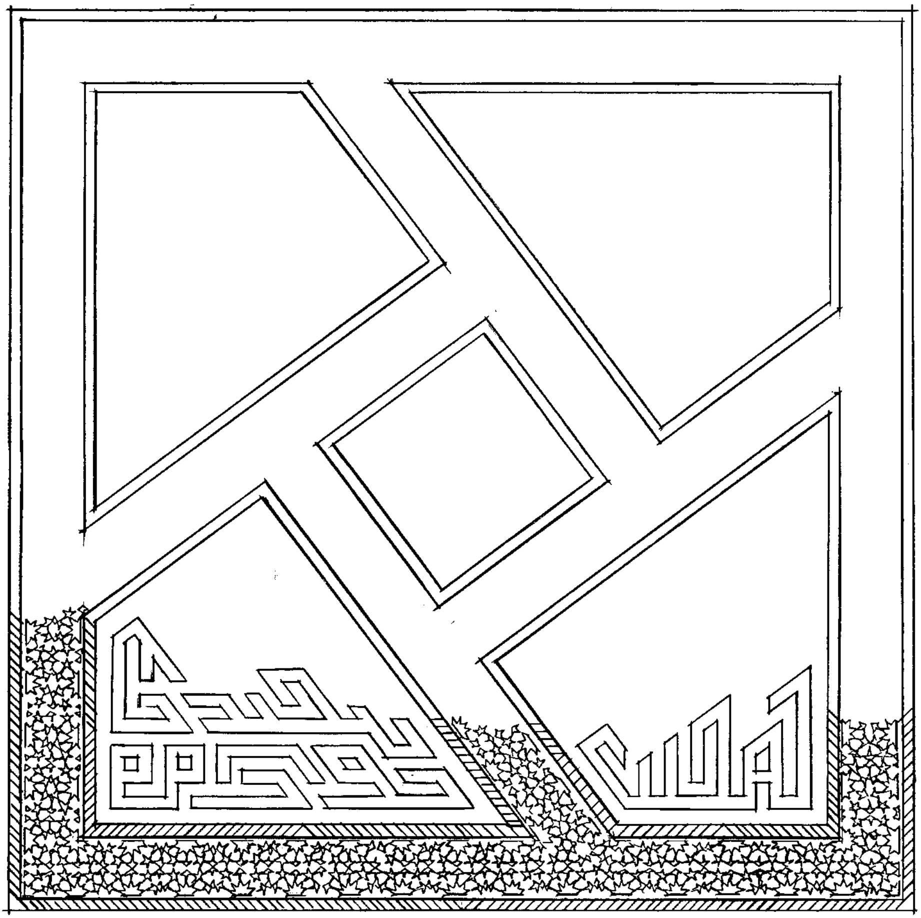


FIG. 2. A 17th-century tile panel from the western *iwan* of the Friday Mosque in Isfahan.

If we write a for the side of the unit square and x for the side of the constructed square, the construction is equivalent to $x = a(1 + \frac{1}{2}\sqrt{2})$. For practical purposes, the construction produces a tolerable approximation of the exact value $\tilde{x} = a\sqrt{3}$, with error 1.4%, but it does not possess the qualities necessary for ornamental purposes.

Abu'l-Wafā' continued with a more detailed explanation for the benefit of the artisans:

> He may imagine it to be correct because of the correctness of the angles and the equality of the sides. The angles of the square are correct, each of them is a right angle, and the sides are equal, and because of this it is imagined to be correct. The reason is as follows. Each of the angles of the [three] triangles, namely G, B, and D which are [also] angles of the square, is a right angle; and the fourth angle is composed of two angles each of which is half a right angle, for they are the angles of the trapezia. The sides are straight and equal, since each of these sides is composed of a side of one of the squares plus half its diagonal that are equal. It is also clear that they are straight [lines] when assembled, because the sums of the angles at the meeting points of the lines are all equal to two right angles.

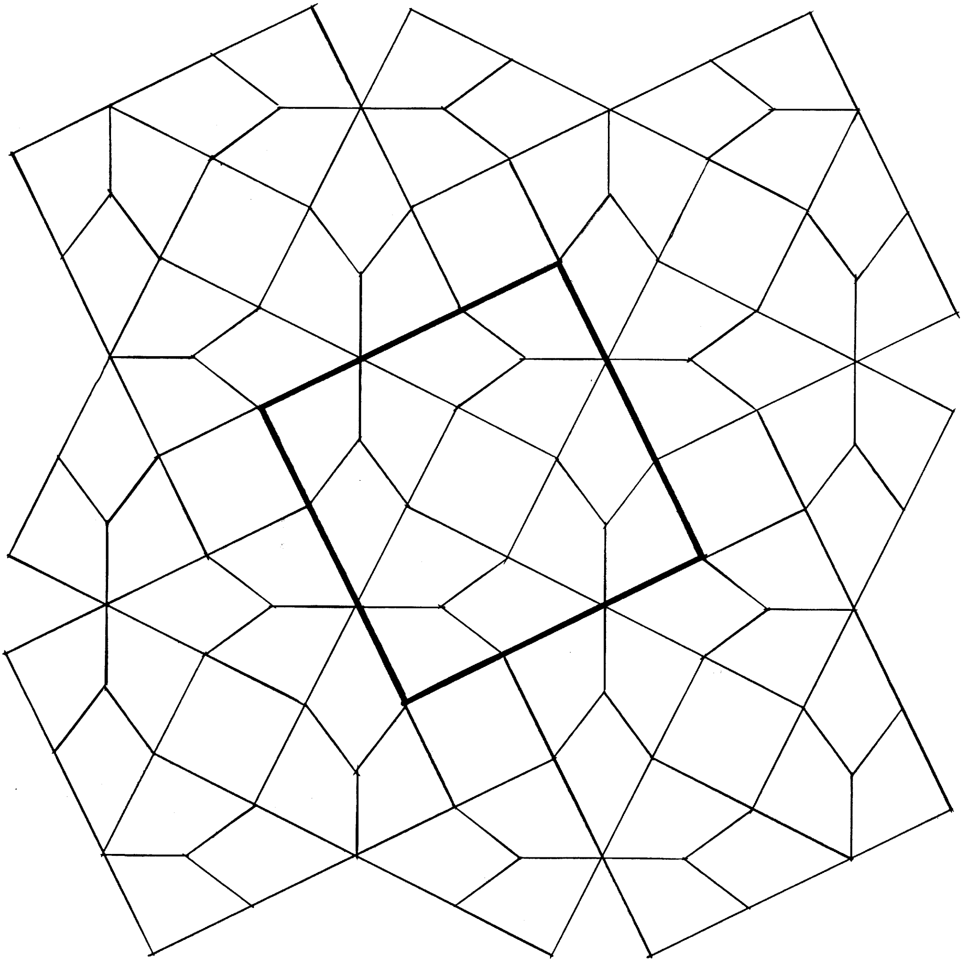


FIGURE 3

> For the three angles at point H are equal to two right angles, since they are one angle of a square and two angles of a triangle, each of which is equal to half a right angle. The same [is true] for angle T. Angle I consists of two angles; one of them is the angle of the triangle, that is half a right angle, and the other is the angle of the trapezium, that is one right angle plus half a right angle. The same [is true] for the two angles at point K. Since the angles are right angles and the sides are straight and equal, everybody imagines that it [i.e., the figure] is a square constructed from three squares. [4, 54]

From this, we infer that the figures were drawn on a board of some sort, not actually cut. Had the three unit squares been dissected and the pieces reassembled to form a single square, it would have been obvious that the sides of the trapezia and of the triangles do not meet at points I and K (as shown in Fig. 4 by broken lines), and Abu'l-Wafā' 's task of disproving the construction would have been simpler.

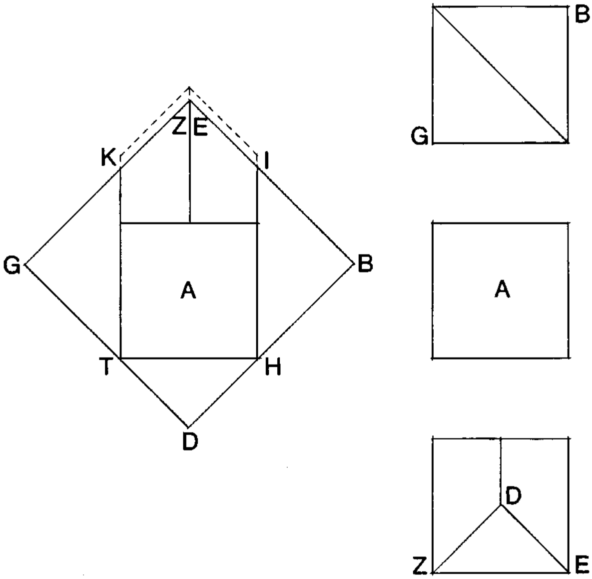


FIGURE 4

Since the error had not become clear, he continued:

> But they do not notice the place where the error and mistake enters their argument. This is clear if we know that each of the sides of this square is equal to the side of one of the [unit] squares plus half of its diagonal. It is not possible that the side of the square composed from three squares has this magnitude, since it must be greater. The reason is as follows. If we make the side of each [unit] square approximately ten ells, to make it easy for the student, the side of the square composed of three squares is by approximation seventeen and one-third ells. But the side of this square [which has been constructed] is seventeen and one-fourth ells,[15] and there is a big difference between them.
>
> Again, as we bisected [the unit] square *BG* and placed each half of it at the side of the square *A*; and thus [the constructed] square *BG* is cut by two lines, *HI* and *TK*, that [the latter] does not coincide with it [the former] because of two things. First, the diagonal of square *BG* is irrational but line *HI* is rational since it is equal to the side of square *BG* plus half of it. Second, it is less than that because the diagonal of square *BG* is by approximation fourteen and one-seventh, and side *HI* is fifteen. Thus the incorrectness of this [method of] division and assembling has become clear. [4, 54–55]

This is the only passage in Abu'l-Wafā''s text where he used numerical approximations of irrational ratios. Constructive geometry could be used when the object was small enough to be constructed by means of a compass and a ruler by an ordinary artisan, such as a mason or a carpenter. Because of the tendency to modular design in Islamic architecture, numerical approximations were more convenient for artisan–architects or surveyors in designing or laying out a building or a pattern based on a geometric scheme.

Root extraction was a standard operation in the arithmetical works of the Islamic world, and also in Greek works on astronomy and trigonometry. The numerical approximations

[15] The approximation $\sqrt{2} \approx 99:70$, which follows from $BG \approx 14\frac{1}{7}$ mentioned below, has as its consequence $x:a = 1+\frac{1}{2}\sqrt{2} \approx 17\frac{1}{14}:10$. It is therefore likely that the value $17\frac{1}{4}$ in the text is somehow the result of a scribal error.

that Abu'l-Wafā' gave may have been common among architect–artisans and surveyors. The approximation[16] $\sqrt{3} = \tilde{x} : a \approx (17 + 1/3) : 10$ occurs frequently in the equivalent form $\sqrt{3} \approx 26 : 15$ in Hero of Alexandria's works on applied geometry [26, passim]. These works were transmitted into Arabic and were important sources of medieval Islamic treatises on surveying [45].

Abū'l-Wafā''s approximation $BG \approx 14\frac{1}{7}$ is equivalent to $\sqrt{2} \approx 14\frac{1}{7} : 10 = 99 : 70$. This approximation does not tally with the approximation $\sqrt{2} \approx 17 : 12$, which Hero commonly used. I therefore suggest the following possible explanation related to the pre-Eudoxean theory of ratios. Two ratios are equal in this theory if, in modern terms, their continued fraction expansions are the same. Youschkevitch [49, 84] and Kennedy [30, 663] pointed out that this theory was studied by medieval Islamic mathematicians. These mathematicians may have approximated irrational ratios by convergents of continued fraction expansions. The fractions $26 : 15$ and $99 : 70$ are the fifth convergents in the continued fraction expansions of $\sqrt{3}$ and $\sqrt{2}$, respectively [24, 129 and 145].[17] Thus, it is conceivable that Abū'l-Wafā' used the equivalent of modern continued fraction expansions to generate his approximations of $\sqrt{2}$ and $\sqrt{3}$. More evidence would be necessary to assure this explanation.

Abu'l-Wafā' continued with the second proposal of the artisans:

> Some people have divided these squares in another way, which is even more clearly incorrect than the first division. This [method] is as follows. In the middle of the diagonal of two of the squares, a segment is cut off equal to one of their sides. From the two endpoints of [this segment of] the diagonal, four triangles are cut off. Thus, two squares become four irregular pentagons and four triangles. Then each pentagon is placed at one side of the third square. At its four angles one obtains [empty] places for four triangles. Then the remaining triangles are transported to these places, and one obtains a square from three squares as in this figure [Fig. 5]. [4, 55]

If we write a for the side of the unit square and x for the side of the constructed square, the construction is again equivalent to $x = a(1 + \frac{1}{2}\sqrt{2})$. The construction may possess some ornamental value, since a simple pattern can be generated by repetition of the constructed squares. Abu'l-Wafā' then pointed out that the main cause of the error is the confusion between the sides and the hypotenuses of the small triangles:

> Someone who does not have experience in geometry and proofs may imagine this to be correct, but if one examines it, it is clearly false and incorrect. The reason is that the triangles that were transported to the empty places at the angles of the square are greater than the places. For each of the empty places is contained by two sides and a diagonal. Each side [of the empty place] is equal to half the hypotenuse of the triangle cut off from the square; so this is its side and its chord is equal to the hypotenuse [of the triangle], and this is absurd.
>
> Example [i.e., explanation in specific notation] of this: We make one of the triangles *ABG* and one of the pentagons *AEZHD*. If the pentagon is transported to the sides of the square, and the triangles to their places, point *G* of triangle *ABG* coincides with point *G* of the square, and [side] *AG* of the triangle coincides with [side] *AD* of the pentagon. But [side] *AD* of the pentagon is equal to [segment] *AD* of the triangle, which is half the hypotenuse. Thus, half of the hypotenuse of the right-angled triangle is equal to its side, that is *AG*. This is absurd and impossible. [4, 55–56]

Abu'l-Wafā' next discussed the error once more by means of the regular octagon formed by cutting off two more right-angled triangles from the remaining angles of the unit square.

[16] Abu'l-Wafā' often used unit fractions in his works, particularly in his text on practical arithmetic for scribes and other officials [44, passim].

[17] The fraction $17 : 12$ is the third convergent of $\sqrt{2}$.

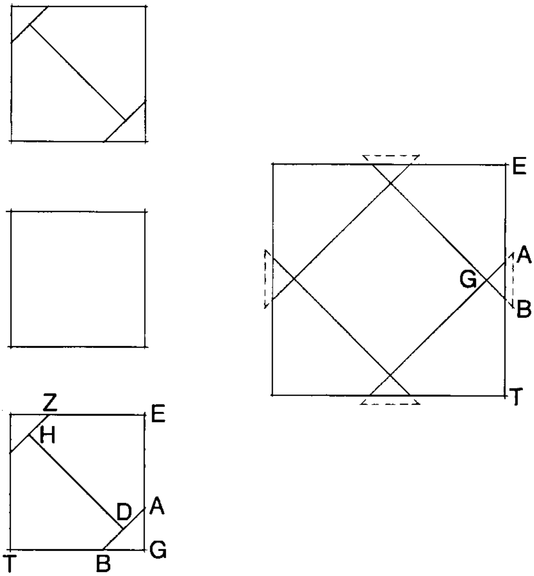


FIGURE 5

Putting a for the side of the octagon and following Abu’l-Wafā’’s argument by inserting modern mathematical expressions gives the following:

> Again, AB [a] is the side of the [regular] octagon inscribed in the square, and AE plus BT is equal to two times the side of the octagon plus the excess of the side of the square over he side of the octagon [i.e., $AE + BT = 2a + a\sqrt{2}$]. Thus line ET [of the constructed square] is equal to three times the side of the octagon plus the excess of the side of the square over the side of the octagon [$ET = AE + AB + BT = 2a + a\sqrt{2} + a = 3a + a\sqrt{2}$]. This is also absurd, for the side of the square composed from three squares is much less than this. Thus the incorrectness of what they constructed has become clear, as we have mentioned in this chapter. [4, 56]

This completed Abu’l-Wafā’’s explanation of the mistakes in the artisans’ constructions. His patience, tolerance, conscientiousness, and teaching skills indicate that his title “professor” or “sage” was well-deserved.

He next presented his own solution:

> The division of the squares by the correct method, as required by proof, will be clearer according to the method that we now mention. We bisect two of the squares along their diagonals. Each of those is applied to one side of the third square: we place one of the angles of the triangle which is half a right angle at one angle of the square, and the hypotenuse of it [the triangle] at the side [of the square]. Thus, part of the triangle sticks out at the other angle [of the square]. Then we join the right angles of the triangles by means of straight lines. That becomes the side of the desired square. From each big triangle, a small triangle is cut off for us [by a straight line], and we transfer it to the [empty] position of the triangle that is produced at the other angle.
>
> Example of this: If we want to construct a square from three equal squares *ABGD*, *EWZH*, *TIKL*, we bisect two of the squares at their diagonals, by means of lines *AG*, *EH*, and we transport [them] to the sides of the [third] square. Then we join the right angles of the triangles by lines *BZ*, *ZW*, *WD*, *DB*. On either side [of the straight line], a small triangle has now been produced from the sides of the [two big]

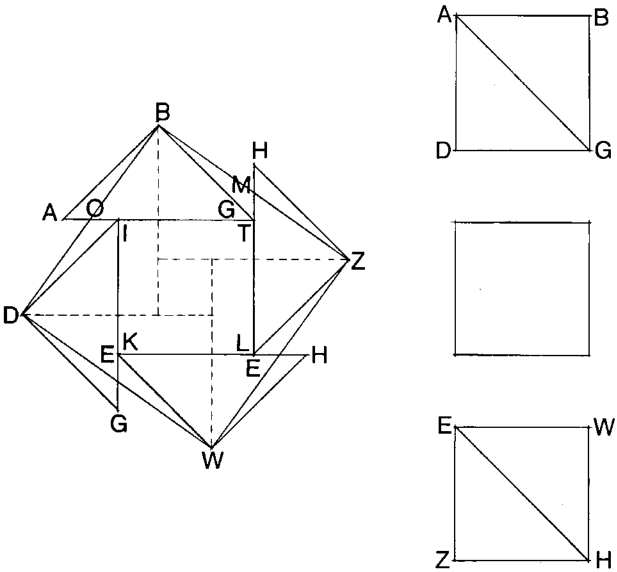


FIGURE 6

> triangles. That [empty position of the triangle] is equal to the triangle which has been cut off from the big triangle. Thus triangle *BGM* is equal to triangle *MZH*, since angle *G* is half a right angle, angle *H* is half a right angle, the two opposite angles of the triangles at *M* are equal, and side *BG* is equal to side *ZH*. Therefore, the remaining sides of the triangles [*BGM*, *MZH*], and the triangles are equal. Thus, if we take triangle *BGM* and put it in the position of triangle *MZH*,[18] line *BZ* is the side of the square constructed from three squares. This is a correct method, easier than what was constructed [by the artisans], and the proof of it has been established. This is the figure for it [Fig. 6]. [4, 56–57]

The fact that Abū'l-Wafā''s construction is based on cut-and-paste methods was certainly attractive for the artisans. The resulting figure is again composed of four triangles rotating around a square at the middle (shown by broken lines in Fig. 6A), but, unfortunately, the solution does not possess the ornamental qualities of the previous construction of a square equal to five squares.

Abu'l-Wafā' concluded his discussions on constructing a square from three squares by explaining the theoretical construction by a geometer that he mentioned at the beginning:

> If the geometer is asked for a construction of a square equal to any number of squares, he will find for you the line that is equal in square to these squares, and he will not be concerned with the way in which the squares have to be cut. That is to say, if he is asked for the construction of a square from three [equal] squares, he will draw the diagonal of one of the squares, and erect at one of the endpoints of the diagonal a line perpendicular to it and equal to the side of the square. He will join its endpoint with the [other] endpoint of the diagonal by means of a straight line. Then that is the side of the square composed of three [equal] squares. Example of this: If we want to construct one square equal to three squares, each of which is equal to square *ABGD*, we draw diagonal *AD*. Then *AD* is the side of the square composed

[18] It would be more correct to say, "if we take triangle *MZH* and put it in the position of triangle *BGM*." There are more problems like this in the text, and the two manuscripts which have been consulted, the Istanbul and Uppsala manuscripts, have disturbing differences here. These problems can perhaps be resolved by consulting medieval Persian translations.

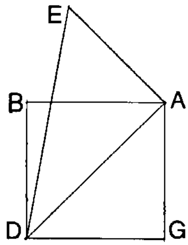


FIGURE 7

> of two squares. Then we erect at point *A* of line *AD* perpendicular *AE* equal to line *AG*, and we join *ED*. Then line *ED* is the side of the square equal to three squares, each of which is equal to square *ABGD* [Fig. 7].
>
> If the geometer has obtained this line, he is not concerned any more with the way in which the squares have to be cut. He will say that if a square is constructed on line *ED*, it is equal to three squares. And [we can construct] similarly, if we want the square to be equal to more than three squares or less than three squares. [4, 57–58]

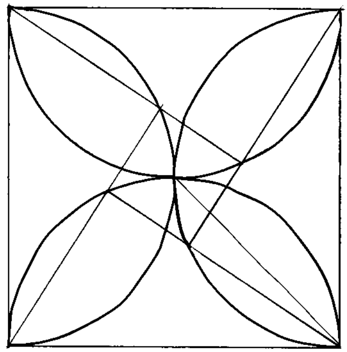
On the whole, the discussions on the construction of a square from three squares did not offer much material which the artisans could use for creating new ornamental patterns.

However, the next chapter, titled "On Dividing One Square into a Number of Squares Which [Number] Is Not the Sum of Two Square Numbers," dealt with the inverse problem. There, Abu'l-Wafā' presented another figure that can be used for ornamental purposes. The figure belongs to a preliminary construction: from a given big square to cut off a smaller square with a given side such that the remaining area is also a square.[19] To effect this (Fig. 8A) describe four semicircles on four sides of the big square. Using a compass with one arm on the angular point of the big square and compass opening equal to the side of the smaller square, mark four points on the semicircles as in the figure, and join these points by straight lines to the four angular points. Thus, one obtains four right-angled triangles and another square in the middle. In these right-angled triangles, the hypotenuse and the shortest side correspond to the sides of the larger and smaller squares, respectively, and the intermediate side becomes the side of the required square [4, 59–60].

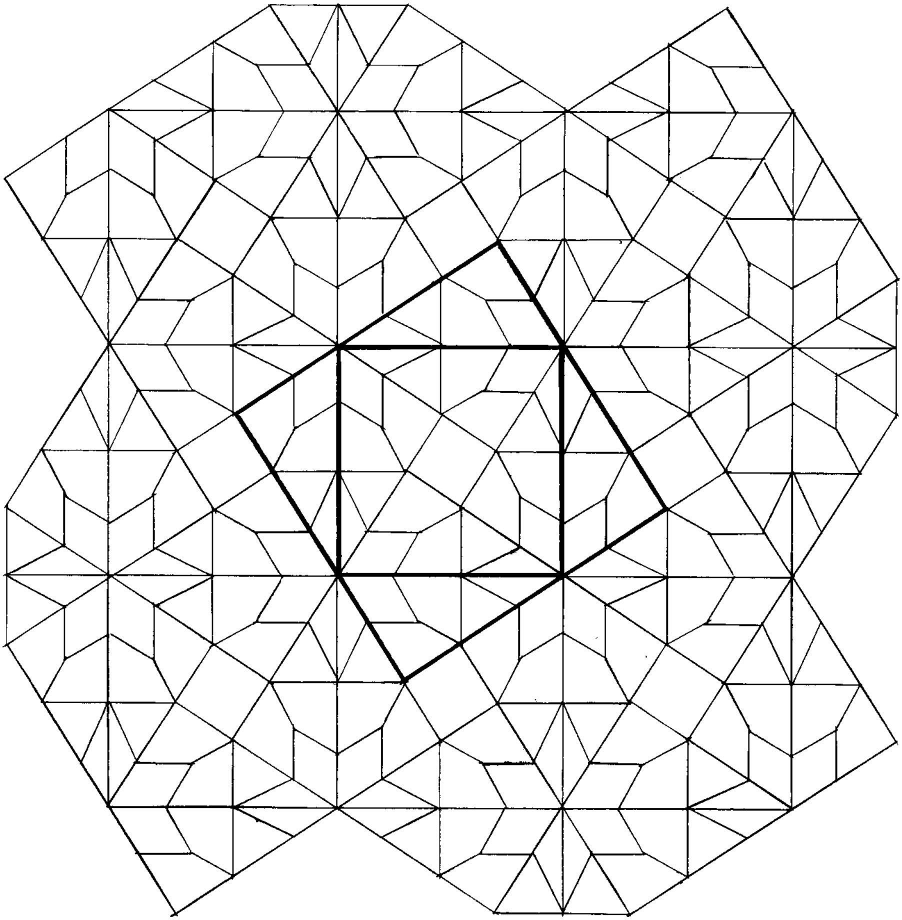
Abu'l-Wafā''s figure contains four right-angled triangles rotating around a square in the middle, as in his earlier constructions of a square equal to five or three squares. This figure underlies many patterns in extant architectural monuments. Although the artisans could choose any ratio they wanted between the sides of the triangle, in most of the motifs that I have studied, the ratio was $1:2$, corresponding to the construction of a square from five squares, but some displayed other ratios. The latter group indicates that the artisans utilized Abu'l-Wafā''s figures in a variety of decorative motifs.

More importantly, Abu'l-Wafā''s figures seem to have paved the way for the creation of another interesting pattern. Around 1074, ʿUmar Khayyām described a special right-angled triangle, in which the hypotenuse is equal to the sum of the short side and the perpendicular to the hypotenuse. He showed that the construction of this triangle is equivalent to the

[19] Abu'l-Wafā' suggested the following relation between this preliminary construction and the problem in the title: Assume a square with side a, which has to be divided into n unit squares. Choose two numbers p and q such that $n = p + q$. Choose a smaller square with side b such that $b^2 : a^2 = p : n$ and solve the preliminary problem for this smaller square. He did not explain the construction of b.



A



B

FIGURE 8

solution of a cubic equation, solved it by the intersection of conic sections, and also offered an approximate calculation [39, 59–64]. Around 1300, a special case of Abu'l-Wafā''s figure reappeared in *Interlocks of Figures* with rotating triangles of the type described by ʿUmar Khayyām. There, a verging construction (i.e., *neusis* construction) of this pattern was offered, plus four more approximate solutions with varying degrees of accuracy [39, 64–67]. Repetition of this pattern generates a delicate composition of squares, triangles, almonds, and stars (Fig. 8B), which was inspired by the elegance of Abu'l-Wafā''s figure.

INTERLOCKS OF FIGURES

The text of *Interlocks of Figures* was composed in the Ilkhānid period of Persia. After the Mongol invasion in 1220, Persia suffered decades of catastrophic destruction and almost complete architectural inactivity. The revival of architecture began slowly under Hulagu (1245–1265). A notable example was the construction of the observatory at Marāgha (ca. 1258) under the supervision of the celebrated astronomer Naṣīr al-Dīn al-Ṭūsī. In this period, Tabrīz became the capital of the Ilkhānids. It was, however, during the reign of Arghun (1284–1291) that Tabrīz began its evolution into a rich commercial and cultural center with splendid suburbs. The golden age of the Ilkhānids began with the reign of Ghazan Khān (1295–1304). He and his vizier, Rāshid al-Dīn (1247–1318), undertook huge construction campaigns around Tabrīz and gathered a great number of scholars, scientists, and artisans there. Rāshid al-Dīn specifically dedicated his suburb, the Quarter of Rāshid, to the encouragement of the arts and sciences. This architectural activity continued during the reign of Öljeitü (1304–1316), who started and completed the construction of Sulṭāniye to replace Tabrīz as the capital.

From this brief historical account of Ilkhānid development, Tabrīz suggests itself as the most likely place of origin of *Interlocks of Figures*. The outburst of creative architectural energy there would naturally have brought artisans and geometers into intimate collaboration and could easily have led to the creation of *Interlocks of Figures*.

The following construction in *Interlocks of Figures* provides insight into the connection between this work and *Geometric Constructions*, and into the unchanging nature of the collaboration between artisans and geometers.

> Section: if we want to transform rectangle *ABGD* into a square, it is first required that on the extension of line *AG* we draw line *GE* equal to *GD*. We bisect line *AE* at point Z. We make point Z the center, and, with compass-opening *ZA*, we draw arc *EHA*. Then on the extension of line *GD*, we draw line *GH* until it meets the circumference of circle *EHA*. Thus, line *GH* is, potentially, the side of the square. Then from along side *AG*, we cut off magnitude *GT* from point *G* equal to *GH*; from side *BD*, we cut off magnitude *BI*[20] from point B equal to *GH*. We bisect the remainder of the side on either end at points L, K, and we join line *LK*. Then at points T and I, we draw perpendiculars *TM* and *IN*[21] until they meet diagonal *LK*. Four pieces are [thus] produced, and we transform these four pieces into square *STHG* [Fig. 9A]. The construction is completed according to a narrative (taqrīr). God knows best. [11, 182v]

The author of this construction must have been a geometer who used a cut-and-paste method to provide artisans with an alternative to a theoretical construction. In this case, the construction is from Euclid's *Elements*, II:14 [25, 1:409], which is the geometric equivalent

[20] The manuscript text has *BD*.
[21] The manuscript text has *ID*.

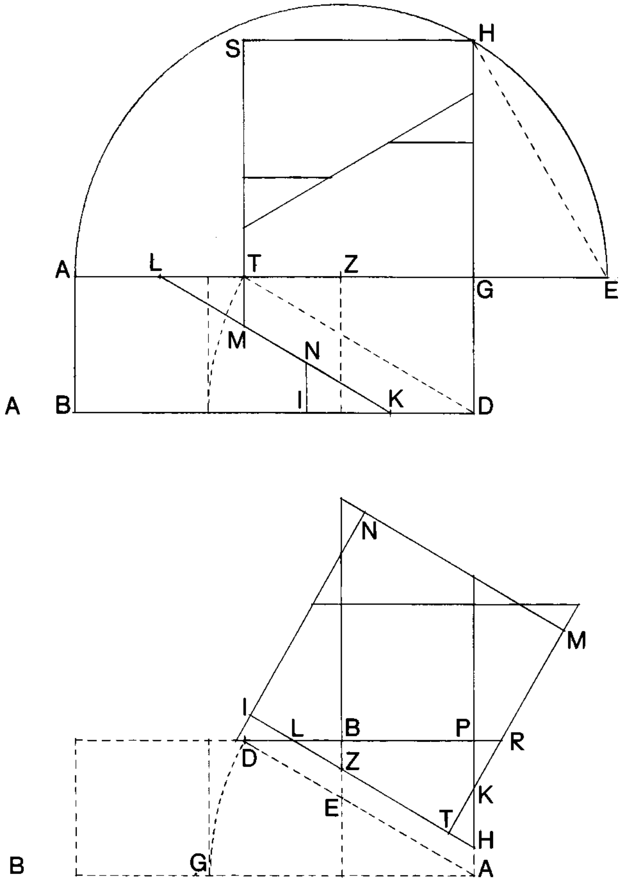


FIGURE 9

of the solution of the algebraic equation $x^2 = ab$ and the extraction of the square root (by taking *GD* as the unit length). The idea of the method is to shift the straight line *DT*, which is equal to *EH*, parallel to the position of *KL* so as to divide rectangle *ABGD* into two halves. Then two congruent triangles, *IKN* and *TLM*, are taken out so that they fit in the empty spaces when the two halves are slid along the dividing line to form the square. This alternative construction is easy but too simplistic to be used for ornamental purposes.

If the ratio between the short and long sides of the rectangle is 1 : 3 (as shown in Fig. 9A), we obtain an alternative construction of a square equal to three unit squares. In this case, line *DT* corresponds to the side of a square equal to two unit squares. Essentially the same construction (Fig. 9B) and the construction by Abū’l-Wafā’ (Fig. 6) appear side by side in another treatise on practical geometry and applied mensuration, *The Work of Abū Bakr al-Khalīl al-Tājir al-Raṣadī* (A’mal-i Abū Bakr al-Khalīl al-Tājir al-Raṣadī). The constructions are unrelated to the text of this work, but they appear in the margin in two drawings with a note: “Two alternative solutions for the problem of composing a square from three squares” [9, 136v]. The equivalence of Figs. 9A and 9B suggests that Abū Bakr al-Khalīl was the author of the construction of Fig. 9A as well. He was the only mathematician cited in *Interlocks of Figures* as one of the participants at the discussions with artisans. We can assume that Abū Bakr al-Khalīl was present when the problem of transforming a rectangle into a square was discussed at one of the meetings. He must have seen that the proposed construction (Fig. 9A) could be applied to the specific problem of composing a square from three squares in the way of Fig. 9B so as to create an ornamental pattern. The text by Abū Bakr al-Khalīl does not give any explanation for his construction of Fig. 9B.[22]

Abū Bakr al-Khalīl's construction is not as elegant as that of Abu'l-Wafā'. However, repetition of the figure of Abū Bakr al-Khalīl's construction generates an interesting composition that would have appealed to the artisans (Fig. 10). After the meeting at which the transformation of a rectangle into a square was discussed, Abū Bakr al-Khalīl may have worked out his construction for artisanal use. To record this contribution to the ornamental arts, he added the two constructions to his treatise that he was working on at the time.

Most of the first seven manuscript leaves of *Interlocks of Figures* concern the transformation from one polygon to another by cut-and-paste methods. In the early sessions documented in *Interlocks of Figures*, the mathematicians seem to have chosen a method of teaching that might be useful for ornamental purposes as well. The mathematicians may thus have used the cut-and-paste methods to explain new theoretical concepts to the artisans. Some of the artisans, on the other hand, may well have considered the method to be an amusing procedure by which to create puzzle-like games, and they were not overly worried about the difference between theoretical exactness and approximation. The first two constructions of *Interlocks of Figures* illustrate the two contrasting attitudes of mathematicians and artisans toward geometry that still existed three centuries after Abu'l-Wafā'.

In the first construction, the text explains how one large decagon can be put together from two unit decagons and a pentagonal star that fits in the space between the decagons:

> Whenever two congruent regular decagons are drawn, [in between fits] a pentagonal seal (khātam-i mukhammasī) of which the sides correspond to the sides of the decagon. Of this [pentagonal seal], the external diameter is equal to the external diameter of the decagon, and half the internal diameter is equal to the side of the decagon. These three figures can be [divided and then] assembled in one decagon. This is the decagon of which the sides are equal to half the external diameter of the assumed decagon and of which half the external diameter is equal to half the external diameter of the assumed decagon plus its side, as is illustrated [Fig. 11].[23] [11, 180r]

The term "pentagonal seal" is used in the text for the special pentagonal star that ties together

[22] A modern explanation with proof can be given as follows. Figure 9B displays the three unit squares in a horizontal row. Let a be the side of the unit square. Point D is constructed such that $AD = 2a$. Then $BD = a\sqrt{3} - a$. The angles in triangle DEB are 30°, 60°, and 90°; hence $BE = a - a/\sqrt{3}$, $ED = 2a - 2a/\sqrt{3}$. Therefore $BD + DE + BE = 2a$. Let Z be the midpoint of BE and choose H as in the figure such that $HA = BZ$. Draw line HZ and extend it to meet BD at L. Then triangles BED and BZL are similar so $BZ + ZL + LB = a$. Choose T on HZ such that $TH = BZ$ and draw TK perpendicular to TH. Then triangle TKH is congruent to triangle BLZ. Therefore $TK + KH + TH = TK + KA = a$, so $TK = PK$. Extend TK to meet point line LP at point R. Then triangles RPK and HTK are congruent.

Since line HZ bisects the unit square, quadrilateral $BZTR$ is also half the unit square. We can divide the other half of the unit square in the same way and put the pieces together as a quadrilateral with angular points at P, R, and M as in the figure. If we dissect another unit square in a similar way, we obtain four congruent quadrilaterals which can be placed around the third square as suggested by Fig. 9B. (Point I is defined on TL extended such that $LI = TK$.) Then TI is the side of a square composed of three unit squares.

[23] The broken lines in Fig. 11 represent the uninked guiding lines incised on paper of the original figure.

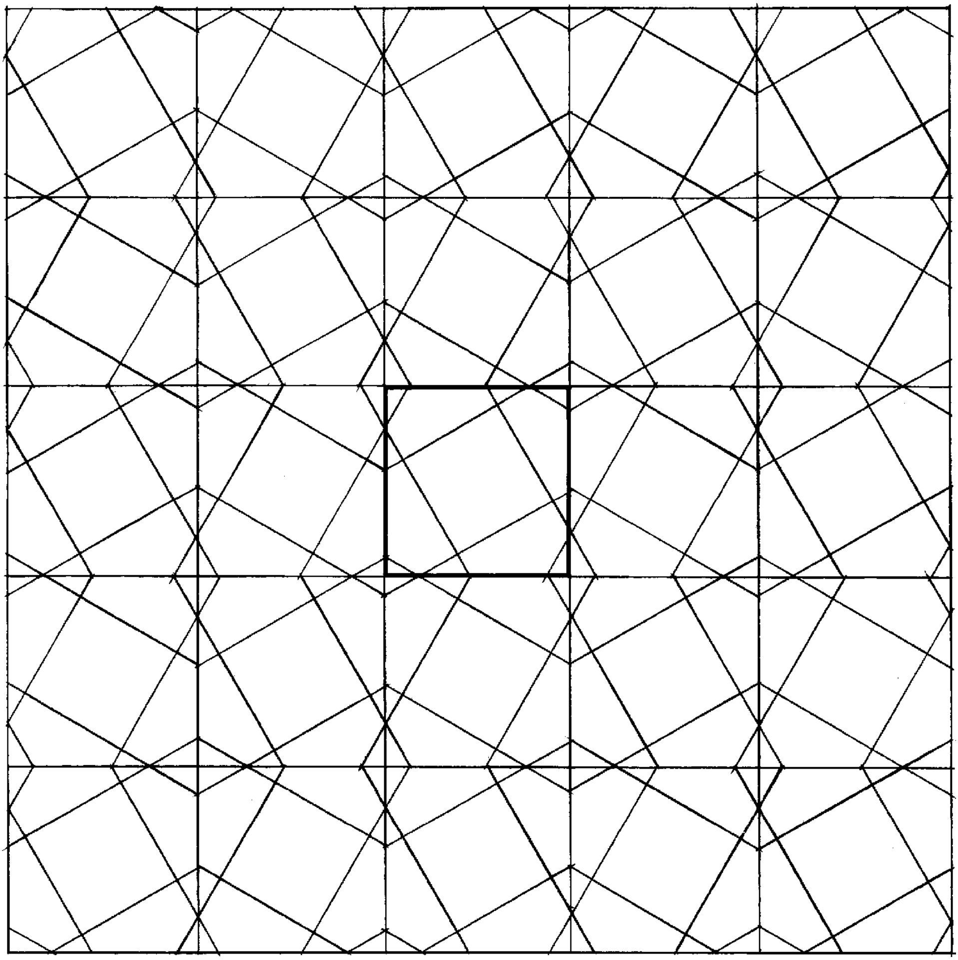


FIGURE 10

the surrounding decagons, to distinguish it from the regular pentagonal star formed by extending the sides of the regular pentagon. The text makes no mention of the fact that the ratio between the radius and the side of the decagon is equal to the ratio of the parts of a segment divided in extreme and mean ratio.[24] For the sake of clarity, we will introduce the notation $\Phi = (\sqrt{5}+1)/2$ for the ratio of the larger to the smaller part of a segment divided in extreme and mean ratio. The construction in the text is thus based on the property that the ratio between the areas of the pentagonal seal and the regular decagon inscribed in the same circle is $1 : \Phi$. By means of this construction, it is easy to show that $\Phi^2 = 2 + 1/\Phi$ because the ratio of the areas of the two decagons is the ratio of the squares of their sides, that is, Φ^2.

[24] See Euclid's *Elements* XIII:9 and compare [25, 3:455]. In *Geometric Constructions*, Abu'l-Wafā' made no mention of division in extreme and mean ratio either, although he based all his constructions of the decagon and the pentagon on this concept.

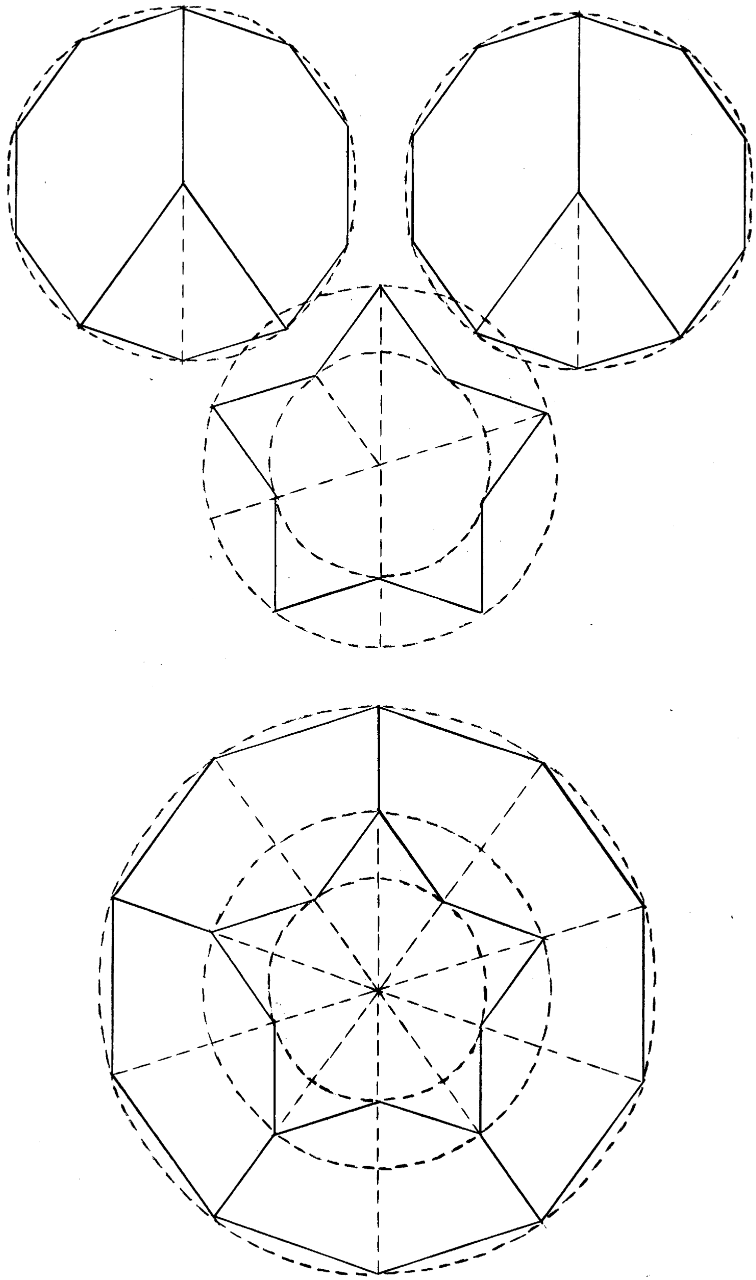


FIGURE 11

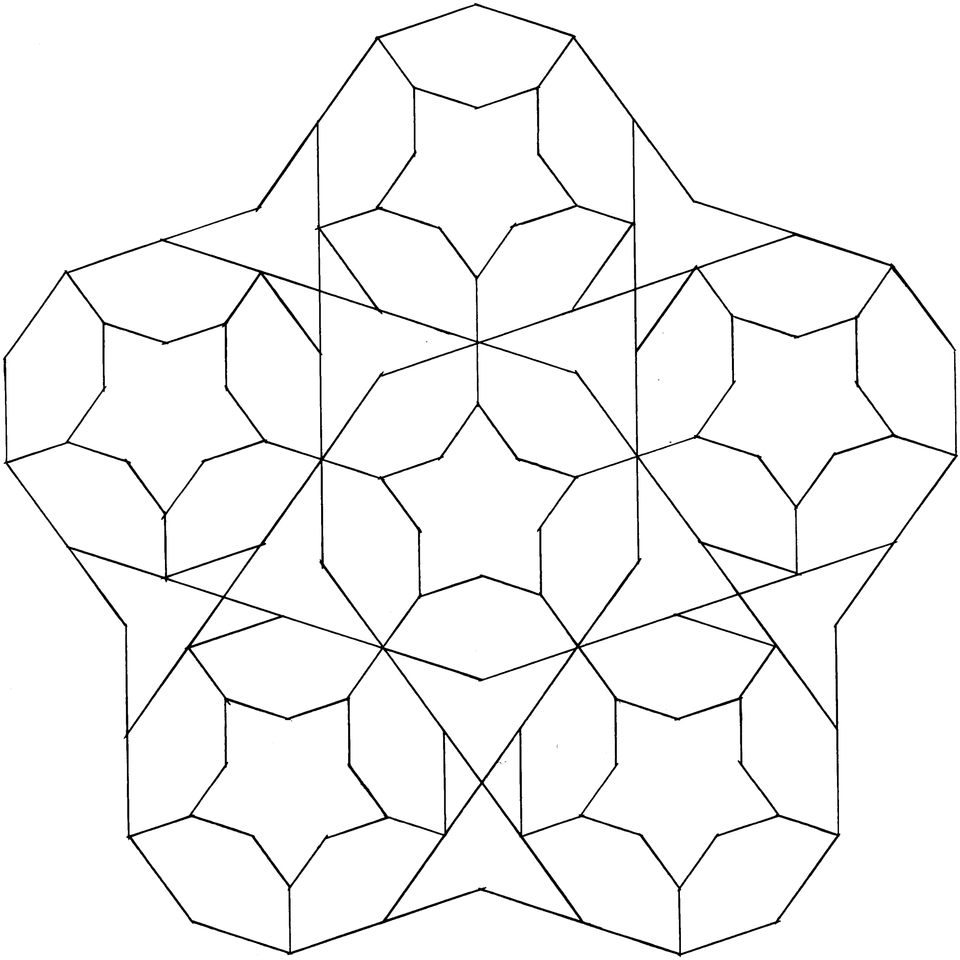


FIGURE 12

The following hypothetical scenario seems possible. A mathematician, proud of his discovery of $\Phi^2 = 2 + 1/\Phi$, could have discussed it at a meeting with the artisans. To demonstrate the proof in a concrete way to the artisans and to attract their attention to its tessellation possibilities, he introduced the construction and used the cut-and-paste method. If the figure is repeated radially a pattern results (Fig. 12).

This meeting and others that followed were recorded and compiled by someone identified elsewhere in the text as a scribe. It is reasonable to suppose that the following construction in the text represented the response by the artisans to the mathematician's construction:

> From the pieces *W* and *Z* altogether one can obtain 1 decagonal star, or if one wants, one can make 2 decagons. Again from the pieces *E* and *D*, 1 nonagonal star can be obtained, or else 2 nonagons. Again from the components *G* and *B*, 1 octagonal star can be obtained, or else 2 octagons. Similarly,

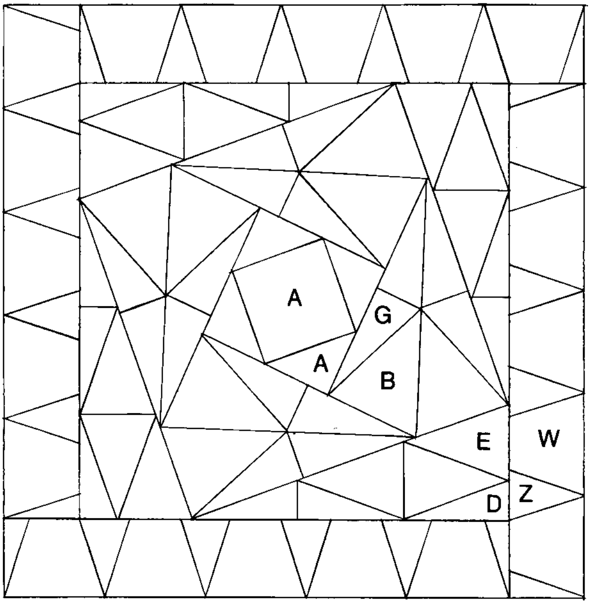


FIGURE 13

from the components *A*, 1 square star or 2 squares can be obtained, as [is shown] in this example [Fig. 13].

It must be known that from the components of this [large] square, 1 square star, 1 octagonal star, 1 nonagonal star, and 1 decagonal star can be obtained. If one wants 2 squares, 2 octagons, 2 nonagons, and 2 decagons can be composed. This [construction] is extremely delicate [laṭīf]. God knows best. [49, 180v]

The author of this construction called the square composed of two squares a "square star." I know of no other occurrence of this term in Arabic or Persian mathematical texts, and a mathematician would call it a square. The construction is not complete because the author did not explain how the large square should be cut into components so as to assemble them in the stated polygons. An analysis of the figure reveals that the construction starts by dividing each side of the large square into 23 equal parts. Three of these parts determine the thickness of the outermost belt, and the remaining 20 are cut into five pieces W, four pieces Z, and two pieces equal to half of Z. When one piece W and one piece Z are combined at their equal sides, one obtains the radial triangle of the proposed decagon. The angle of the vertex of this triangle is equal to 36.87° since the tangent of the half angle is equal to $1/3$. This is in error by about 2.4% from the correct value, 36°. Ten of these triangles are meant to form a decagon (Fig. 14).[25] It is unclear why such an approximate decagon was formed from a square belt. The construction seems neither to serve a didactic purpose nor to have any ornamental value. The approximative construction of the decagon recalls similar artisanal constructions described by Abu'l-Wafā' in *Geometric Constructions* and suggests that the author of the construction of Fig. 13 was also an artisan.

[25] Figure 14 does not appear in the original text.

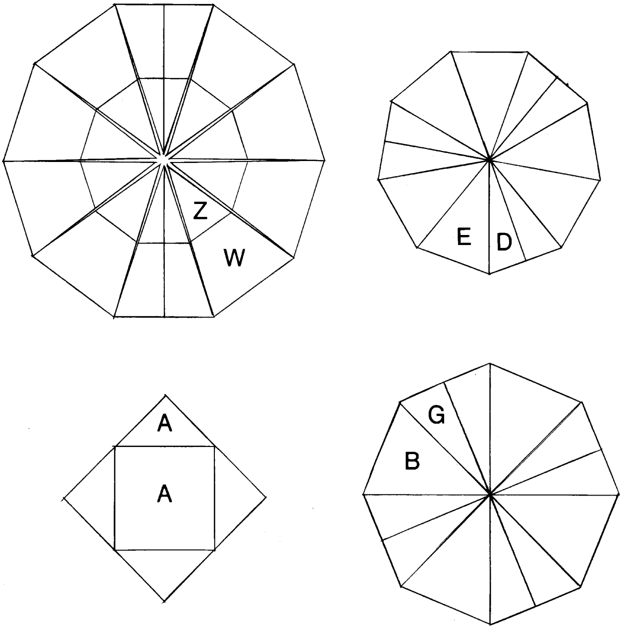


FIGURE 14

Inside the square belt are placed four rotating right-angled triangles, which are subdivided into pieces to be assembled as a nonagon. To make the construction exact, the angles at the vertices of the triangular pieces *D* and *E* should be equal to 20° and 40°. A correct construction of these angles presupposes the trisection of an angle of 60°, which can be performed with theoretical exactness by means of conic sections or verging procedures. In *Geometric Constructions* Abu’l-Wafā’ proposed two different verging procedures for the trisection of the angle and advised one of them for the construction of a nonagon [4, ff. 17, 23, 30]. These methods would be cumbersome here, and since the construction of the decagon was approximate, an approximation method would be expected here as well, in line with the attitude of the artisans toward geometry.

The pieces *B* and *G* to be used for the octagon form four triangles rotating around the square composed of two squares at the middle. The arrangement of these four rotating triangles shows that the artisan was familiar with the pattern based on Abu’l-Wafā’’s construction mentioned above. If the four triangles are constructed so that their smallest angles are 22.5°, the triangles can be divided and assembled into a regular octagon. In the central part of the figure, the cut-and-paste method is theoretically correct and has ornamental potential.

On the whole, the solution is incoherent and inconsistent and can hardly be the work of a mathematician. The problem seems to have arisen as a challenge to the practically minded artisans. The artisan who worked out the puzzle-like construction was evidently so proud of his complicated solution that he described it as “extremely delicate.”

CONCLUSION

We have little information on what artisans of the Islamic world knew about geometry, but we learn from several sources that mathematicians, throughout the centuries, taught the basic principles of geometry to artisans and architects. In this brief study of *Geometric Constructions* (ca. 998) and *Interlocks of Figures* (ca. 1300), I have argued that mathematicians used cut-and-paste methods in their teaching of geometry for two purposes: to prove the correctness of certain constructions in a concrete way that could be easily understood by the artisans, and to present the constructions in such a way that the figures could be used to create new decorative patterns. In the hands of some talented mathematicians, the method turned out to be very effective and led to the discovery of a number of new patterns that became quite popular. Thus the artisans and architects of the Islamic world benefited from the cooperation and expertise of these mathematicians, who enriched Islamic art and architecture.

The same artisans, however, did not seem to use the opportunity to develop their own knowledge of geometry. Although Abu'l-Wafā' wrote his *Geometric Constructions* especially for that purpose, the *Interlocks of Figures* shows that artisans were still dealing with geometry in their unmethodical and incorrect way three centuries after Abu'l-Wafā'. The reason may have been that artisans always had access to the advice of mathematicians at their meetings, and therefore never took the trouble to acquire the knowledge themselves. Caʿfer Efendi reported that a mathematician who taught geometry to artisans of the royal court in Istanbul still complained in 1570 about their ignorance of geometry. His words suggest that not much had changed in the six centuries after Abu'l-Wafā''s time: "Regarding that which is called the science of geometry, in this age, if the science of geometry is discussed among architects (miʿmar) and learned men (ʿālim), each one [of the architects] will answer, 'Yes, we have heard of it, but in essence we have not heard how the science of geometry works and what it deals with'" [16, 28].

APPENDIX: THE ARABIC TEXT

The Arabic text in this appendix is based on the following two manuscripts of *Geometrical Constructions* by Abu'l-Wafā' [46, 324]: Istanbul, Aya Sofya 2753 ff. 52–58, (see the facsimile in [42]), and Uppsala, No. 324, ff. 47b–52b. The Uppsala manuscript has often been misattributed to al-Fārābī (compare [46, 296]).

The sole purpose of this Appendix is to provide the Arabic text on which the translations in the paper are based. There are numerous small differences between the two Arabic manuscripts, and for the sake of brevity, a critical apparatus has not been provided here. In cases where the two manuscripts are different, the reading of the Istanbul manuscript has usually been chosen, unless this manuscript is clearly defective. The passages between angular brackets $\langle \cdots \rangle$ are missing in the Istanbul manuscript but have been supplied from the Uppsala manuscript.

A critical edition of the whole treatise of Abū'l-Wafā' with English translation would be a most welcome addition to the literature. Such an edition should also take account of the medieval Persian tradition of the work.

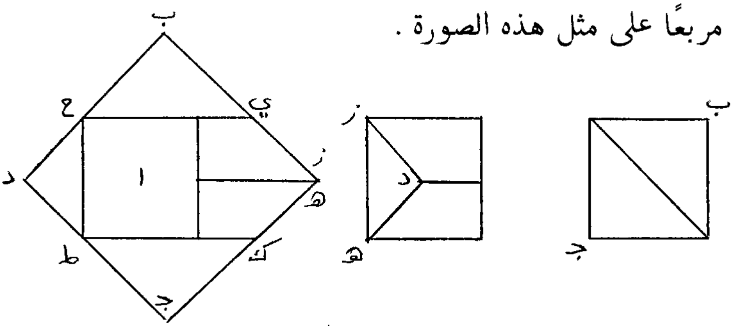
في تركيب المربعات وقسمتها اذا لم يكن عددها مؤلفًا من عددين مربعين .

قد غلط جماعة من المهندسين والصنّاع في امر هذه المربعات وتركيبها . اما المهندسون فلقلة دربتهم بالعمل واما الصناع فلخلوهم من علم البراهين . وذلك أن المهندسين اذا لم تكن لهم دربة بالعمل صعب عليهم تقريب ما يصح له بالبراهين الخطوطية على ما يلتمسه الصانع . فإن الصانع غرضه ما يقرب عليه العمل ويظهر له صحة ما يراه في الحس والمشاهدة ولا يبالي بالبراهين الخطوطية والمهندس اذا أقام له البرهان على الشيء بالتوهم لم يبال صحة ذلك بالمشاهدة او لم يصح . على أنا لا نشك أن جميع ما يراه الصانع إنما هو مأخوذ مما يعمله المهندس أولًا وقام البرهان على صحته فإن الصانع والماسح إنما يأخذ من الشيء زبدته ولا يفكر في الوجوه التي يثبت صحة ذلك به ، ولأجل ذلك قد يقع عليه الغلط والخطأ . فأما المهندس فقد علم صحة ما يريد بالبراهين اذا كان هو المستخرج للمعاني التي عمل عليها الصانع والماسح ، وانّما < يصعب عليه ردّ ما يعمله بالبرهان الى العمل اذا لم يكن له دربة بما يعمله الصانع ولماسح > فإن حذاق هؤلاء المهندسين اذا سئلوا عن شيء من قسمة الاشكال او شيء من ضرب الخطوط تحيّروا فيه واحتاجوا الى فكر طويل ، وربما سنح لهم هذا وقرب عليهم ، وربما صعب ولم يتأت لهم عمله .

ولقد حضرت في بعد المجالس وفيه جماعة من الصناع والمهندسين وسئلوا عن عمل مربع من ثلاث مربعات . اما المهندس فانه استخرج خطًا يقوى على ثلاث مربعات بسهولة ولم يرض احد من الصنّاع < بما عمله > . فان الصانع يريد ان يقسم تلك المربعات بأقسام يؤلف منها مربعًا واحدًا < كما عملنا في مربعين وخمس مربعات > . واما الصنّاع فانهم اوردوا فيها عدّة وجوه قام البرهان على البعض وبطل البعض الا ان ما لم يقم البرهان عليه كان مقارنًا للصحّة في المنظر فيتخيّل لمن

ينظر اليه أنه صحيح . ونحن نورد تلك الوجوه ليعلم الصحيح منه من الفاسد ولا يقع على الناظر في هذا المعنى غلط في قبول ما هو فاسد ان شاء اللّه .

وذلك ان بعض الصناع وضع احد المربعات في الوسط وقطع الثاني منها بنصفين بالقطر ووضعه عن جنبتي المربع واخرج من وسط الثالث الى زاويتين منه على غير القطر خطين مستقيمين وخطًا منه الى وسط الضلع المقابل للمثلث < الذي > حدث بالخطين فانقسم المربع بمنحرفين ومثلث ثم وضع المثلث اسفل المربع الاول ووضع المنحرفين فوقه وضمّ الضلعين الاطولين احدهما الى الآخر في الوسط فصار له مربعًا على مثل هذه الصورة .



قال ابو الوفاء اما صورة ما عمله فهو في التخيل ومن لا تكون له دربة بالصناعة والهندسة يرى انه صحيح واذا كشف عنه علم انه خطأ .

اما انه يتوهم انه صحيح فمن جهة صحّة الزوايا واستواء الاضلاع فان زوايا المربع صحيحة كل واحدة منها قائمة . واما الاضلاع فانها متساوية ولأجل هذا يتخيل انه صحيح . وذلك ان زوايا مثلثات جَ بَ دَ التي هي زوايا المربع كل واحدة منها قائمة والزاوية الرابعة مركبة من زاويتين كل واحدة منهما نصف قائمة وهما زاويتا المنحرفين . واما الاضلاع فمستقيمة ومتساوية وذلك ان كل واحد من هذه الاضلاع مركّب من ضلع احد المربعات ومن نصف قطره فهي متساوية . وأما انها مستقيمة في التركيب فهو بين ايضًا فان الزوايا المجتمعة عند التقاء الخطوط كلها

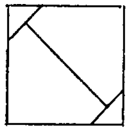
مساوية لقائمتين لان الثلاث الزوايا التي عند نقطة ح مساوية لقائمتين لانها زاوية مربع وزاويتا مثلث كل واحدة منهما نصف قائمة وكذلك زاوية ط . واما زاوية ي فانها زاويتان احداهما زاوية المثلث وهي نصف قائمة والاخرى زاوية المنحرف وهي قائمة ونصف وكذلك الزاويتان اللتان عند نقطة ك . واذا كانت الزوايا قائمة والاضلاع مستقيمة متساوية يتخيّل لكل واحد انها مربعة عملت من ثلاث مربعات .

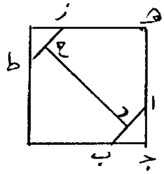
ولا يفطنون للموضع الذي دخل عليهم < الخطأ و > الغلط < منه > وانّما تبين ذلك انا قد علمنا ان كل ضلع من اضلاع هذا المربع قد صار مساويًا لضلع احد المربعات ولنصف قطره . فليس يجوز ان يكون ضلع المربع المؤلف من ثلاث مربعات هذا المقدار فانه اكثر منه . وذلك ان ضلع المربع المؤلف من ثلاث مربعات اذا جعلنا ضلع كل مربع عشرة اذرع تقريبًا على المتعلم هو سبعة عشر ذراعًا وثلث بالتقريب وضلع هذا المربع هو سبعة عشر ونصف سبع وبينهما تفاوت كثير .

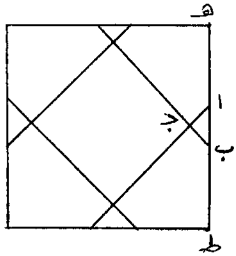
وايضًا فان مربع بج لما قسمناه بنصفين ووضعنا كل نصف منه الى جانب مربع آ وقطع مربع بج على خطي حي طك وليس يجوز ان يقع عليه ذلك لشيئين . احدهما ان قطر مربع بج لا ينطق به وخط حي منطق وهو مثل ضلع مربع بج ومثل نصفه والثاني انه اصغر منه وذلك ان قطر مربع بج هو اربعة عشر وسبع بالتقريب وضلع حي هو خمسة عشر فقد تبين فساد هذه القسمة والتركيب .

وقد قسم بعض الناس هذه المربعات بنوع آخر من القسمة اظهر فسادًا من القسمة الاولى . وذلك انه فصل من قطر مربعين منها في وسطه مثل احد ضلعها وقطع من طرفى القطر اربع مثلثات فيصير المربعان اربع مخمسات مختلفات الاضلاع واربع مثلثات . ثم وضع كل مخمس الى ضلع المربع الثالث فيصير في اربع زواياه موضع اربع مثلثات فنقل المثلثات الباقية اليها فصار مربعًا مؤلفًا من ثلاث مربعات

على هذه الصورة .







وهذا ايضًا يتخيل لمن لا يكون له دربة بالهندسة والبرهان انه صحيح ومتى تأمل ظهر انه فاسد خطأ . وذلك ان المثلثات التي نقل الى المواضع الفارغة من زوايا المربع هي اكبر من مواضعها وذلك ان المواضع الفارغة يحيط بكل واحد منها ضلعان وقطر كل ضلع منها مساو لنصف قطر المثلث الذي قطع من المربع فهذا ضلعه < ووتره > مساو للقطر وهذا محال . مثال ذلك انا نجعل احد المثلثات عليه ابج وأحد المخمسات اهزحد فاذا نقل المخمس الى اضلاع المربع والمثلثات الى مواضعها وقعت نقطة ج من مثلث ابج على نقطة ج من المربع ووقع اج من المثلث على اد من المخمس لكن اد من المخمس مساو لـ اد من المثلث وهو نصف الوتر فصار نصف الوتر من المثلث القائم الزاوية مساو لضلعه وهو اج وهذا محال لا يجوز .

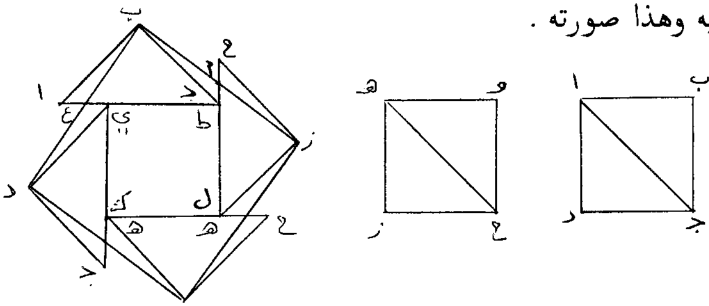
وايضًا فان اب هو ضلع المثمن الذي يقع في المربع و اه و بط هو مساو لمثلي ضلع المثمن ولفضل ضلع المربع على ضلع المثمن فيصير خط هط مساويًا لثلاثة امثال ضلع المثمن ولفضل ضلع المربع على ضلع المثمن وهذا ايضًا محال . فان ضلع المربع المؤلف من ثلاث مربعات هو اقل من هذا بكثير . فقد تبين فساد ما عملوه كما ذكرنا في هذا الفصل

فامّا قسمة المربعات على الوجه الصحيح وعلى ما يلزم عليه البرهان فانه يتبين على الوجه الذي نذكره . وهو انا نقسم مربعين منها بنصفين على قطره ويضاف كل واحد منها الى ضلع من اضلاع المربع الثالث ونجعل الزاوية التي هي نصف

قائمة من المثلث على زاوية من زوايا المربع والقطر منه على الضلع فيفضل لنا من المثلث من عند الزاوية الاخرى بعضه . ثم نوصل بين زوايا المثلثات القائمة بخطوط مستقيمة فيكون ذلك ضلع المربع المطلوب وينفصل لنا من كل مثلث كبير مثلث صغير وننقله الى موضع المثلث الحادث عند الضلع الآخر

مثال ذلك انا اردنا ان نعمل من ثلاث مربعات متساويات وهي مربعات ابجد هوزح طيكل < مربعًا > قسمنا مربعين من المربعات بنصفين نصفين على قطريهما بخطي اج هح ونقلناها الى اضلاع المربع ثم وصلنا بين الزوايا القائمة من المثلثات بخطوط بز زو ود دب وحدث في كل جانب من اضلاع المثلث مثلث صغير مساو للمثلث الذي انفصل من المثلث الكبير فصار مثلث بجم مساو لمثلث مزح لان زاوية ج نصف قائمة وزاوية ح نصف قائمة والزاويتان المتقابلتان من المثلثين عند م متساويتان وضلع بج مساو لضلع حز فصار باقي اضلاع المثلثات مساويًا لباقي الاضلاع والمثلث مساو للمثلث

فاذا اخذنا مثلث بجم ووضعنا في موضع مثلث مزح < صار > خط بز ضلع المربع المعمول من ثلث مربعات . وهذا هو وجه صحيح اقرب مما عمل قد قام البرهان عليه وهذا صورته .

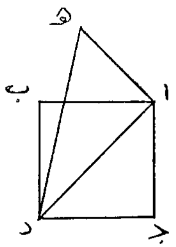


فاما المهندس فانه اذا سئل عن عمل مربع من مربعات قلّت او كثرت فانه يوجدك الخط الذي يقوى على تلك المربعات ولا يبالي بتقطيع المربعات كيف كانت . وذلك انه اذا سئل عن عمل مربع من ثلاث مربعات فانه يوصل قطر احد المربعات ويقيم على احد طرفي القطر خطًا يكون عمودًا عليه مساويًا لضلع المربع

ويوصل بين طرفه وبين طرف القطر بخط مستقيم فيكون ذلك ضلع المربع المؤلف من ثلاث مربعات

مثال ذلك اذا اردنا ان نعمل مربعًا واحدًا مساويًا لثلاث مربعات كل واحد منها مساو لمربع ابجد اخرجنا قطر اد فيكون اد ضلع < المربع > المركب من مربعين ثم اقمنا على نقطة ا من خط اد عمود اه مساويًا لخط اج ووصلنا هد فيكون خط هد ضلع المربع المساوي لثلاث مربعات كل واحد منها مساو لمربع ابجد

فاذا حصل عند المهندس هذا الخط لم يبال بعد ذلك كيف كان تقطيع المربعات وقال انه متى عمل على خط هد مربعًا كان مساويًا للمربعات الثلاثة وكذلك لو اردنا ان يكون المربع مساويًا لاكثر من ثلاث مربعات او اقل منها .



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